(A CLASS OF) HODGE DUALITY OPERATORS
OVER THE QUANTUM SU(2)

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ABSTRACT. On the exterior algebra over $\mathrm{SU}_q(2)$ coming from the four dimensional bicovariant calculus à la Woronowicz we introduce, using sesquilinear contraction maps, a class of metrics and Hodge duality operators, and compare this formulation with the previous ones presented in the literature.

CONTENTS

1. INTRODUCTION

This paper is a further step along the study of the geometry that the quantum SU(2) group acquires, when it is equipped with the $4D_+$ first order bicovariant calculus à la Woronowicz and the corresponding (not universal, four dimensional) exterior algebra $\Omega(\mathrm{SU}_q(2))$. It is aimed to evolve the analysis presented in [11], where, using the canonical braiding of the calculus, a definition of Hodge operators on both $\Omega(\mathrm{SU}_q(2))$ and $\Omega(\mathrm{S}_q^2)$ (with S_q^2 the standard Podleś sphere) were proposed. The squares of those Hodge operators do not show the classical degenerate spectra: they are not constant on each k-form in $\Omega^k(\mathrm{SU}_q(2))$, where they turn out to be not even necessarily diagonal.

Is it possible to introduce novel Hodge operators on $\Omega(\mathrm{SU}_q(2))$, whose square is diagonal, and such that its spectrum has a degeneracy which naturally fits with the degeneracy of the antisymmetriser operators (used to construct the exterior algebra) coming from the canonical (non trivial) braiding associated to the $4D_+$ first order calculus?

To answer this question, we merge the formulation described in [11] with (again!) that in [5], and introduce a notion of Hodge operators on $\Omega(\mathrm{SU}_q(2))$ whose square is diagonal, with a spectrum

showing the *same* degeneracy of the antisymmetrisers of the calculus. Such Hodge operators and their corresponding metric structures present close similarities to those studied in [5], as well as interesting differences with those in [11] and in [3, 4], where a Hodge map on $\Omega(\mathrm{SU}_q(2))$ has been introduced via a suitable ϵ tensor.

1.1. **Overview.** It is well known what Hodge duality operators on classical group manifolds are. Consider an N-dimensional real connected Lie group G. A metric on the group G is a non degenerate tensor $g : \mathfrak{X}(G) \otimes \mathfrak{X}(G) \to \mathcal{A}(G)$ (with $\mathcal{A}(G) = \mathrm{Fun}(G)$ the $*$-algebra of complex valued coordinate functions on G) which is symmetric, that is $g(X,Y) = g(Y,X)$ for any pair of vector fields $X, Y \in \mathfrak{X}(G)$, and real, $g^*(X,Y) = g(Y^*, X^*)$. Written in its normal form on a real basis $\{\theta^a = \theta^{a*}, a = 1, \ldots, N\}$ of $\Omega^1(G)$, it is

$$g = \sum_{a,b=1}^{N} \eta_{ab}\, \theta^a \otimes \theta^b, \tag{1.1}$$

where $\eta_{ab} = \pm 1 \cdot \delta_{ab}$. The inverse metric tensor $g^{-1} = \sum_{a,b=1}^{N} \eta^{ab} X_a \otimes X_b$ (with $\sum_b \eta^{ab}\eta_{bc} = \delta_c^a$ on the dual vector field basis such that $\theta^b(X_a) = \delta_a^b$) allows to define an $\mathcal{A}(G)$-bilinear contraction map $\tilde{g} : \Omega^{\otimes k}(G) \times \Omega^{\otimes s}(G) \to \Omega^{\otimes |k-s|}(G)$ given by

$$\tilde{g}(\theta^{a_1} \otimes \ldots \otimes \theta^{a_k}, \theta^{b_1} \otimes \ldots \otimes \theta^{b_s}) = \left(\Pi_{i=1}^k \eta^{a_i b_i} \right) \theta^{b_{k+1}} \otimes \ldots \otimes \theta^{b_s}, \tag{1.2}$$

for $k < s$. In order to set such a map on the exterior algebra $\Omega(G)$, we recall that exterior forms are defined as the range of the classical antisymmetriser operators (those constructed from the braiding given by the flip),

$$\theta^{a_1} \wedge \ldots \wedge \theta^{a_s} := A^{(s)}(\theta^{a_1} \otimes \ldots \otimes \theta^{a_s})$$
$$= (A^{(k)} \otimes A^{(s-k)})\, B_{k,s-k}(\theta^{a_1} \otimes \ldots \otimes \theta^{a_s}), \tag{1.3}$$

with $B_{k,s-k} := \sum_{p_j \in S(k,s-k)} \mathrm{sign}(p_j)\, p_j : \Omega^{\otimes s}(G) \to \Omega^{\otimes s}(G)$ the operator obtained in terms of the shuffles $S(k, s-k) \subset P(s)$, namely those permutations p_j of s elements such that $p_j(1) < \ldots < p_j(k)$ and $p_j(k+1) < \ldots < p_j(s)$. The relation above proves that the expression

$$\tilde{g}(\theta^{a_1} \wedge \ldots \wedge \theta^{a_k}, \theta^{b_1} \wedge \ldots \wedge \theta^{b_s}) := \tilde{g}(A^{(k)}(\theta^{a_1} \otimes \ldots \otimes \theta^{a_k}), A^{(s)}(\theta^{b_1} \otimes \ldots \otimes \theta^{b_s})) \tag{1.4}$$

consistently defines a contraction $\tilde{g} : \Omega^k(G) \times \Omega^s(G) \to \Omega^{s-k}(G)$. Fix an ordering $\vartheta = \theta^1 \otimes \ldots \otimes \theta^N$ of the basis so to have the volume N-form $\mu = \mu^* := A^{(N)}(\vartheta)$ (an orientation). The corresponding Hodge duality $\star : \Omega^k(G) \to \Omega^{N-k}(G)$ is the operator whose action is

$$\star(\xi) := \frac{1}{k!}\, \tilde{g}(\xi, \mu)$$
$$= A^{(N-k)}(\tilde{g}(\xi, B_{k,N-k}\vartheta)). \tag{1.5}$$

It satisfies the identity

$$\star^2(\xi) = \mathrm{sgn}(g)(-1)^{k(N-k)}\xi \tag{1.6}$$

on any $\xi \in \Omega^k(G)$. Here $\mathrm{sgn}(g) = \det(\eta_{ab})$ is the signature of the metric.

If the metric is in addition left-invariant, that is the 1-forms θ^a in (1.1) are left-invariant, the volume μ defines a left Haar measure on the group manifold, which allows to introduce a (left-invariant) sesquilinear inner product $\langle\ ,\ \rangle : \Omega^k(G) \times \Omega^k(G) \to \mathbb{C}$ by $\langle \zeta, \xi \rangle := \int_\mu \tilde{g}(\zeta^*, \xi)$; the Hodge duality associated to a left-invariant metric can then be equivalently defined by the implicit equation

$$\langle \zeta, \xi \rangle := \int_\mu \zeta^* \wedge (\star \xi) \tag{1.7}$$

for any pair of k-forms ζ, ξ.

The problem of defining a Hodge duality operator on exterior algebras associated to first order covariant differential calculi à la Woronowicz over quantum groups has been studied following two different approaches, which parallel the two classical definition described above.

A very general path is developed in [5, 6, 7]. Given a $*$-Hopf algebra $\mathcal{H}$ equipped with a suitable bicovariant N-dimensional calculus, the theory of bicovariant bimodules allows, using the corresponding canonical braiding σ (and its inverse σ^-), to introduce a real σ-metrics (which generalises, as an homomorphism of $\mathcal{H}$-bimodules, the concept of real metric tensor on a classical group) and a natural hermitian volume form. Using the (order k) antisymmetriser operators $\mathfrak{A}_\pm^{(k)}$ associated to $\sigma^\pm$, such metric structures extend to contractions between higher order forms in the exterior algebras $\Omega_\pm(\mathcal{H})$; Hodge duality operators $\star_{L,R}^\pm : \Omega_\pm^k(\mathcal{H}) \to \Omega_\pm^{N-k}(\mathcal{H})$ are then given by (left, resp. right) $\mathcal{H}$-linearly contracting an exterior form with the volume. The quantum analogue of the classical (1.6) reads in this formulation

$$\star_{L,R}^\pm \star_{L,R}^\mp (\xi) = \xi \tag{1.8}$$

on any $\xi \in \Omega_\pm(\mathcal{H})$.

A notion of left-invariant sesquilinear inner product on a large class of left-covariant exterior algebras $\Omega(\mathcal{H})$ over a $*$-Hopf algebra $\mathcal{H}$ with an Haar state is introduced in [9], together with a natural definition of Hodge operator. This paper shows how specific constraints among inner products on k- and $(N-k)$-forms give an Hodge operator whose square has the classical (with $\mathrm{sgn}(g) = 1$) spectrum (1.6). This formulation has been also extended in [11, 15] to study Hodge operators on the left-covariant $\Omega(\mathrm{S}_q^2)$ with S_q^2 the standard Podleś sphere.

The aim of the present paper is to introduce novel Hodge duality structures on the exterior algebra coming from the $4D_+$ first order bicovariant calculus à la Woronowicz on $\mathrm{SU}_q(2)$, so to evolve the analysis started in [11, §3], from which we get the geometrical setting and the notations. The quantum group $\mathrm{SU}_q(2)$ and its exterior algebras $\Omega_\pm(\mathrm{SU}_q(2))$ are presented in the appendix §B: the nontrivial spectral resolution of the braiding $\sigma^\pm$ reflects into a nontrivial spectral resolution of the corresponding antisymmetriser operators $A_\pm^{(k)}$ (i.e. $A_\pm^{(k)}(\xi) = \lambda_\xi^\pm \xi$, with $\lambda_\xi^\pm \to k!$ in the formal classical limit $q \to 1$), whose actions do not moreover commute with the $*$-conjugation.

This suggests to consider, in section §2, sesquilinear (instead of bilinear) contraction maps $\Gamma_\pm$ on the vector spaces $\Omega_{\pm\mathrm{inv}}(\mathrm{SU}_q(2))$ of left-invariant forms and to define, explicitly depending on the spectra of the antisymmetrisers, Hodge operators $\mathfrak{T}^\pm, : \Omega_{\pm\mathrm{inv}}^k(\mathrm{SU}_q(2)) \to \Omega_{\pm\mathrm{inv}}^{4-k}(\mathrm{SU}_q(2))$ which generalise the second line out of (1.5) to the quantum setting.

We restrict our analysis to contractions having a $\mathrm{U}(1)$-coinvariance, and we relate the hermitianity and the reality of the contraction (i.e. a compatibility with the braiding and $*$-conjugation) to the behaviour of the corresponding Hodge operators. This approach parallels [11], and somehow reverses the line described in [5], where the properties of a metric tensor give sufficient conditions for the Hodge duality to satisfy (1.8). We shall here indeed study how natural requirements on the spectra of the Hodge operators $\mathfrak{T}^\pm$ will constraint the family of contractions $\Gamma_\pm$ from which they are defined.

A contraction $\Gamma_\pm$ will be called maximally hermitian provided the spectra of the square of the corresponding Hodge operators have the same degeneracy of the antisymmetrisers; the main result of this analysis is that the Hodge operators $\mathfrak{T}^\pm$ corresponding to a real maximally hermitian contraction Γ satisfy the identities, on the above diagonal basis for the antisymmetrisers,

$$(\mathfrak{T}^\pm)^2(\xi) = (-1)^{k(4-k)}(\mathrm{sgn}\,\Gamma)\,\frac{\lambda_\xi^\pm}{\lambda_{\xi^*}^\pm}\,\xi,$$

$$\mathfrak{T}^\pm\,\mathfrak{T}^\mp(\xi) = (-1)^{k(4-k)}(\mathrm{sgn}\,\Gamma)\,\xi \tag{1.9}$$

3

with a suitable definition of the signature of Γ: this induces to define Hodge dualities $\star_{\pm}^{L,R}$ as the extension of the $\mathfrak{T}^{\pm}$ (*only* for real and maximally hermitian contractions) to the whole exterior algebras $\Omega_{\pm}(\mathrm{SU}_q(2))$ via the requirement of left (or right) $\mathcal{A}(\mathrm{SU}_q(2))$-linearity.

From real maximally hermitian contractions we set (left-invariant) metric structures $\tilde{g}_q \in \mathfrak{G}$, whose signatures turn out to be constant: this is the same happening to the class of bicovariant real σ-metrics introduced on $\mathrm{SU}_q(2)$ in [5, 6]; a comparison between the two notions of metrics will reveal even more interesting similarities: this class of Hodge dualities allows furthermore to recover the quantum Casimir of $\mathcal{U}_q(\mathfrak{su}(2))$ (the envelopping algebra dual to $\mathcal{A}(\mathrm{SU}_q(2))$) as a Laplacian on $\mathrm{SU}_q(2)$.

We close this overview by saying that, to make the paper self-contained, appendix §A recalls basic notions on the theory of differential calculi and exterior algebras on quantum groups.

Conventions and notations. We define a q-number as

$$[u] := \frac{q^u - q^{-u}}{q - q^{-1}} \tag{1.10}$$

for any $q \neq 1$ and $u \in \mathbb{R}$; for a coproduct Δ we use the conventional Sweedler notation $\Delta(x) = x_{(1)} \otimes x_{(2)}$ with implicit summation, iterated to $(\mathrm{id} \otimes \Delta) \circ \Delta(x) = (\Delta \otimes \mathrm{id}) \circ \Delta(x) = x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$, and so on.

2. Hodge duality operators on $\mathrm{SU}_q(2)$

Motivated by the formulation we shall introduce in the quantum setting, we start by describing here how Hodge dualities can be defined in the classical setting in terms of sesquilinear contraction maps. On a N-dimensional real Lie group we call contraction an $\mathcal{A}(G)$-sesquilinear map $\Gamma :$ $\Omega^1(G) \times \Omega^1(G) \to \mathcal{A}(G)$ with $\Gamma(f\,\phi, \phi') = f^*\Gamma(\phi, \phi')$ and $\Gamma(\phi, \phi'\,f) = \Gamma(\phi, \phi')f$ for $f \in \mathcal{A}(G)$, which we extend (see (1.2),(1.3)) to $\Gamma : \Omega^k(G) \times \Omega^s(G) \to \Omega^{s-k}(G)$ via

$$\Gamma(\phi^{a_1} \wedge \ldots \wedge \phi^{a_k}, \phi^{b_1} \wedge \ldots \phi^{b_s}) := \Gamma(A^{(k)}(\phi^{a_1} \otimes \ldots \otimes \phi^{a_k}), A^{(s)}(\phi^{b_1} \otimes \ldots \phi^{b_s})). \tag{2.1}$$

Consider an ordered left-invariant basis $\{\phi^1, \ldots, \phi^N\} \subset \Omega^1(G)$, and define $\vartheta = m\,\phi^1 \otimes \ldots \otimes \phi^N$ with $m \in \mathbb{C}$ so that $\mu = \mu^* = \mathfrak{A}^{(N)}(\vartheta)$ is an hermitian volume form for the calculus. Given $\xi \in \Omega^k(G)$, the expression

$$T(\xi) := (A^{(N-k)}(\Gamma(\xi, B_{k,N-k}\theta)))^* \tag{2.2}$$

defines an $\mathcal{A}(G)$-linear operator $\Omega^k(G) \to \Omega^{N-k}(G)$. In order to recover the operator T as an Hodge operator corresponding to a metric structure, define the tensor $\tilde{g} : \Omega^{\otimes 2}(G) \to \mathcal{A}(G)$ on the above basis as

$$\tilde{g}(\phi^a, \phi^b) := \Gamma(\phi^{a*}, \phi^b); \tag{2.3}$$

it is then immediate to see that $T(1) = \mu^*$ and $T(\mu^*) = (m^2 \det \tilde{g})^*$, where $\det \tilde{g}$ is the determinant of the matrix $\tilde{g}(\phi^a, \phi^b)$.

Imposing on the sesquilinear Γ both an hermitianity and a reality condition will impose, as in [11], both a symmetry and a reality condition on the tensor $\tilde{g}$:

- The sesquilinear contraction map Γ is said hermitian provided the operator (2.2) satisfies

$$T^2(\phi) = (-1)^{N-1}\,T^2(1)\,\phi \tag{2.4}$$

  for any $\phi \in \Omega^1(G)$. Easy calculations using the properties of the antisymmetriser (and shuffle) operators prove that

$$T^2(\phi) = (-1)^{N-1}\,\{T^2(1)\}\,\phi \qquad \Leftrightarrow \qquad \tilde{g}(\phi^a, \phi^b) = \tilde{g}(\phi^b, \phi^a): \tag{2.5}$$

  hermitian sesquilinear contractions Γ corresponds to symmetric bilinear tensors $\tilde{g}$.

- The sesquilinear contraction map Γ is said real provided

$$T(\phi^*) = (T(\phi))^* \tag{2.6}$$

for any $\phi \in \Omega^1(G)$. Such a reality conditions on the sesquilinear map Γ is equivalent to a reality condition for the bilinear tensor $\tilde{g}$, namely

$$\tilde{g}^*(\phi^a, \phi^b) = \tilde{g}(\phi^{b*}, \phi^{a*}), \tag{2.7}$$

and moreover implies that the action of the operator T will commute with the $*$-conjugation, i.e. $T(\xi^*) = (T(\xi))^*$, on any $\xi \in \Omega(G)$.

Upon fixing the scale parameter by $m^2 \det \tilde{g} = \mathrm{sgn}(\tilde{g})$, it is straightforward to see that the operator T associated to hermitian and real contraction maps Γ is the Hodge operator on $\Omega(G)$ corresponding to the inverse of the real metric tensor $\tilde{g}$ given by Γ via (2.3).

Remark 2.1. *We recall, since we shall use it in the following, that the action of the operator T coincides with the action of the operator $L : \Omega^k(G) \to \Omega^{N-k}(G)$ defined in the same setting by*

$$L(\xi) := \frac{1}{k!}\,\Gamma^*(\xi, \mu) \tag{2.8}$$

on $\xi \in \Omega^k(G)$; hermitianity and reality condition are given by:

$$\phi^* \wedge L(\phi') = \Gamma(\phi, \phi')\mu,$$
$$L(\phi^*) = (L(\phi))^* \tag{2.9}$$

on any left-invariant 1-form ϕ.

2.1. **(Sesquilinear) Contraction maps in the quantum setting.** In order to introduce Hodge operators on the exterior algebras $\Omega_\pm(\mathrm{SU}_q(2))$ generated by the antisymmetriser operators $A_\pm^{(k)}$ (as described in appendix §B.1) we shall start by considering left $\mathcal{A}(\mathrm{SU}_q(2))$-invariant contractions as sesquilinear maps

$$\Gamma : \Omega_{\mathrm{inv}}^1(\mathrm{SU}_q(2)) \times \Omega_{\mathrm{inv}}^1(\mathrm{SU}_q(2)) \to \mathbb{C},$$

with $\Gamma(\lambda^*\omega, \omega') = \Gamma(\omega, \lambda\omega') = \lambda\,\Gamma(\omega, \omega')$ for any $\lambda \in \mathbb{C}$ and $\omega, \omega' \in \Omega_{\mathrm{inv}}^1(\mathrm{SU}_q(2))$, which we consistently extend to contractions

$$\Gamma_\pm : \Omega_{\pm\mathrm{inv}}^k(\mathrm{SU}_q(2)) \times \Omega_{\pm\mathrm{inv}}^N(\mathrm{SU}_q(2)) \to \Omega_{\pm\mathrm{inv}}^{N-k}(\mathrm{SU}_q(2)),$$

where the shuffle operators are given by $B_{k,N-k}^\pm := \sum_{p_j \in S(k,N-k)} \mathrm{sign}(p_j)\mathfrak{P}^\pm(p_j)$ in terms of permutation operators $\mathfrak{P}^\pm(p)$ defined by the braiding σ of the bicovariant calculus on $\mathrm{SU}_q(2)$ or of its inverse σ^{-1} (see §A).

From (B.11) we set $\Omega_{\mathrm{inv}}^{\otimes 4}(\mathrm{SU}_q(2)) \ni \vartheta := im\,\omega_- \otimes \omega_+ \otimes \omega_0 \otimes \omega_z$ with $m \neq 0 \in \mathbb{R}$, so to have, from (B.25), $\mu = \mu^* = \check{\mu} = \check{\mu}^*$. On a left-invariant basis in $\Omega_\pm(\mathrm{SU}_q(2))$ which diagonalises (appendix §B.1) the action of the antisymmetrisers,

$$A_\pm^{(k)}(\xi) = \lambda_\xi^\pm\,\xi, \tag{2.10}$$

we define the operator $\mathfrak{T}^\pm : \Omega_{\pm\mathrm{inv}}^k(\mathrm{SU}_q(2)) \to \Omega_{\pm\mathrm{inv}}^{N-k}(\mathrm{SU}_q(2))$ by

$$\mathfrak{T}^\pm(\xi) := \frac{\lambda_{\xi^*}^\pm}{\lambda_\xi^\pm}\,\left(A_\pm^{(N-k)}(\Gamma_\pm(\xi, B_{k,N-k}^\pm\vartheta))\right)^*. \tag{2.11}$$

These expression generalises to the quantum setting the classical (2.2) taking into account the specific spectrum of the quantum antisymmetrisers.

Since we are interested in Hodge dualities which will give Laplacians whose action restricts (B.7) to any $\mathcal{L}_n \subset \mathcal{A}(\mathrm{SU}_q(2))$, throughout the paper we shall analyse the properties of the operators

defined in (2.11), corresponding to the class of left-invariant contractions which are (right) U(1)-coinvariant. Given $\delta_R^{(1)}\omega_a = \omega_a \otimes z^{n_a}$ with $\delta_R^{(1)} : \Omega^1(\mathrm{SU}_q(2)) \to \Omega^1(\mathrm{SU}_q(2)) \otimes \mathcal{A}(\mathrm{U}(1))$ the extension to 1-forms of the U(1)-coaction (B.6), a contraction map is said U(1)-coinvariant provided

$$n_a \neq n_b \qquad \Rightarrow \qquad \Gamma_{ab} = \Gamma(\omega_a, \omega_b) = 0. \tag{2.12}$$

The coefficient of a U(1)-coinvariant contraction map can be written as (in the ordering defined above):

$$\Gamma_{ab} = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \nu & \epsilon \\ 0 & 0 & \xi & \gamma \end{pmatrix} \tag{2.13}$$

where all the parameters are complex numbers.

2.2. The Hodge operator $\mathfrak{T}^+$.

We start a more detailed analysis of the Hodge operator $\mathfrak{T}^+$. From the definition (2.11) one has

$$\mathfrak{T}^+(1) = \mu,$$
$$\mathfrak{T}^+(\mu) = \Gamma_+^*(\mu, \theta) = m^2 (\alpha\,\beta)^* (\nu\,\gamma - \epsilon\,\xi)^* \tag{2.14}$$

while, on the basis of left-invariant 1-forms,

$$\mathfrak{T}^+ \begin{pmatrix} \omega_- \\ \omega_+ \\ \omega_0 \\ \omega_z \end{pmatrix} = i\,m \begin{pmatrix} 0 & \alpha^* & 0 & 0 \\ -\beta^* & 0 & 0 & 0 \\ 0 & 0 & -\nu^* & \epsilon^* \\ 0 & 0 & -\xi^* & \gamma^* \end{pmatrix} \begin{pmatrix} \chi_- \\ \chi_+ \\ \chi_0 \\ \chi_z \end{pmatrix}. \tag{2.15}$$

As a reality condition, we set the classical relation (2.6), which imposes specific constraints on the contraction coefficient Γ_+ (2.13):

$$\mathfrak{T}^+(\omega_a^*) = (\mathfrak{T}^+(\omega_a))^* \qquad \Leftrightarrow \qquad \begin{array}{l} \beta^* = q^2\alpha, \\ (\nu, \epsilon, \xi, \gamma) \in \mathbb{R}. \end{array} \tag{2.16}$$

For a real contraction map the action of the Hodge operator $\mathfrak{T}^+$ on 3-forms is

$$\mathfrak{T}^+(\chi_-) = -im\,q^{-2}\beta^*(\nu\,\gamma - \epsilon\,\xi)\,\omega_+,$$
$$\mathfrak{T}^+(\chi_+) = im\,q^2\alpha^*(\nu\,\gamma - \epsilon\,\xi)\,\omega_-,$$
$$\mathfrak{T}^+ \begin{pmatrix} \chi_0 \\ \chi_z \end{pmatrix} = im \begin{pmatrix} \gamma^2\nu + (\alpha\,\beta)^*((1-q^2-q^{-2})\gamma - 2(q+q^{-1})^2\nu - (q^2-q^{-2})\xi) & (\alpha\,\beta)^*\xi \\ \epsilon^2\xi + (\alpha\,\beta)^*((q^2-q^{-2})\nu + (1-q^2-q^{-2})\epsilon) & (\alpha\,\beta)^*\nu \end{pmatrix} \begin{pmatrix} \omega_0 \\ \omega_z \end{pmatrix}. \tag{2.17}$$

This allows to set the hermitianity conditions,

$$(\mathfrak{T}^+)^2(\omega_a) = -(\mathfrak{T}^+)^2(1)\,\omega_a \qquad \Leftrightarrow \qquad \begin{array}{l} \beta = q^2\alpha, \\ \xi = \epsilon, \\ \epsilon^3 + \alpha\,\beta((q^2-q^{-2})\nu - (q-q^{-1})^2\epsilon) = 0, \\ \gamma^2\nu = \alpha\,\beta((q^2-q^{-2})\epsilon - 2(q+q^{-1})^2\nu - (q-q^{-1})^2\gamma), \end{array} \tag{2.18}$$

which, together with the reality conditions above, give three families of real and hermitian U(1)-covariant contractions which are invertible (that is, $\mathfrak{T}^+(\mu) \neq 0$):

a) the first one is

$$\begin{array}{l} (\alpha \neq 0, \ \beta = q^2\alpha, \ \nu = 0, \ \gamma \neq 0, \ \epsilon \neq 0, \ \xi = \epsilon) \in \mathbb{R}, \\ \epsilon^2 = (q-q^{-1})^2\alpha\,\beta, \\ (q+q^{-1})\epsilon + (q-q^{-1})\gamma = 0; \end{array} \tag{2.19}$$

b) the second one is:

$$(\alpha \neq 0, \ \beta = q^2\alpha, \ \nu \neq 0, \ \gamma \neq 0, \ \epsilon \neq 0, \ \xi = \epsilon) \in \mathbb{R} \tag{2.20}$$

where the third and fourth out of (2.18) are left implicit. These can be solved if $(q - q^{-1})^2\alpha\beta > \epsilon^2$;

c) the third one is:

$$\begin{aligned}
&(\alpha \neq 0, \ \beta = q^2\alpha, \ \nu \neq 0, \ \gamma = 0, \ \epsilon \neq 0, \ \xi = \epsilon) \in \mathbb{R}, \\
&2\,\epsilon^2 = 3(q - q^{-1})^2\alpha\,\beta, \\
&2(q + q^{-1})\nu + (q - q^{-1})\epsilon = 0.
\end{aligned} \tag{2.21}$$

We notice that, since the braiding of the calculus is not the classical flip, hermitianity and reality constraints are not only expressed by linear relations among the coefficients of the contractions.

The action of the Hodge operator $\mathfrak{T}^+$ on left-invariant 2-forms is:

$$\mathfrak{T}^+ \begin{pmatrix} \varphi_+ \\ \kappa_- \end{pmatrix} = -im\,\alpha \begin{pmatrix} q^2\epsilon + (\frac{q^2+1-q^{-2}}{q^2-1})\nu & (\frac{1}{1-q^2})\nu \\ (q^4 + q^2)\epsilon + (q^4 - q^2)\gamma + (\frac{q^6-q^2+1}{q^2-1})\nu & -\epsilon + (\frac{1}{1-q^2})\nu \end{pmatrix} \begin{pmatrix} \varphi_+ \\ \kappa_- \end{pmatrix}, \tag{2.22}$$

$$\mathfrak{T}^+ \begin{pmatrix} \varphi_- \\ \kappa_+ \end{pmatrix} = -im\,\beta \begin{pmatrix} -q^{-2}\epsilon + (\frac{q^{-2}+1-q^2}{q^{-2}-1})\nu & (\frac{1}{1-q^{-2}})\nu \\ -(q^{-4} + q^{-2})\epsilon + (q^{-4} - q^{-2})\gamma + (\frac{q^{-6}-q^{-2}+1}{q^{-2}-1})\nu & \epsilon + (\frac{1}{1-q^{-2}})\nu \end{pmatrix} \begin{pmatrix} \varphi_- \\ \kappa_+ \end{pmatrix}, \tag{2.23}$$

$$\begin{aligned}
\mathfrak{T}^+(\psi_-) = -\frac{im}{1+q^2} &\left(\{ (\frac{1-q^2-q^{-2}}{1-q^{-2}})\nu\,\gamma + (\frac{2-q^2}{1-q^{-2}})\epsilon^2 + (q^2-1)(q^2 + q^{-2} - 1)\alpha\,\beta \}\psi_- \right. \\
&\left. + \{ (\frac{2q^2-1}{1-q^{-2}})\nu\,\gamma + (\frac{-q^4+q^2-1}{1-q^{-2}})\epsilon^2 + (q^2-1)(q^4 + 1 - q^2)\alpha\,\beta \}\psi_+ \right) \\
\mathfrak{T}^+(\psi_+) = -\frac{im}{1+q^2} &\left(\{ (\frac{1-2q^{-2}}{1-q^{-2}})\nu\,\gamma + (\frac{1-q^{-2}-q^{-4}}{1-q^{-2}})\epsilon^2 + (q^{-2}-1)(q^2 + q^{-2} - 1)\alpha\,\beta \}\psi_- \right. \\
&\left. + \{ (\frac{q^2-1+q^{-2}}{1-q^{-2}})\nu\,\gamma + (\frac{q^{-2}-2}{1-q^{-2}})\epsilon^2 + (1-q^2)(q^2 + q^{-2} - 1)\alpha\beta \}\psi_+ \right).
\end{aligned} \tag{2.24}$$

Every family of real and hermitian contractions (2.19)-(2.21) above will clearly give a specific spectrum to the restriction of Hodge operator $(\mathfrak{T}^+)^2$ on Ω^2_{+inv}. Among them, we look for those conditions giving such spectra a specific degeneracy, thus recalling the classical case (1.6), when the spectrum of square of a Hodge operator is totally degenerate.

We define then a contraction Γ_+ *real* and *maximally hermitian* (with respect to the Hodge operators $\mathfrak{T}^+$) provided:

- it is real and hermitian;
- the operator $(\mathfrak{T}^+)^2$ has the same degeneracy of the antisymmetrisers $A_+^{(k)}$, namely the square of the Hodge operator $\mathfrak{T}^+$ is constant on each eigenspace of the antisymmetrisers $A_+^{(k)}$.

Direct calculations show that the U(1)-coinvariant contraction in (2.13) is real and maximally hermitian provided the constraints (2.19) – given by the set of conditions a) above – are fullfilled.

7

The action of the corresponding Hodge operator $\mathfrak{T}^+$ turns out to be, from the expressions (2.22)-(2.24), as:

$$\mathfrak{T}^+(\varphi_+) = -im\,q^2\alpha\,\epsilon\,\varphi_+,$$
$$\mathfrak{T}^+(\kappa_+) = -im\,q^2\alpha\,\epsilon\,\kappa_+,$$
$$\mathfrak{T}^+(\psi_+) = im(q^2-1)\alpha\,\beta\,\psi_+,$$
$$\mathfrak{T}^+(\varphi_-) = im\,\alpha\,\epsilon\,\varphi_-,$$
$$\mathfrak{T}^+(\kappa_-) = im\,\alpha\,\epsilon\,\kappa_-,$$
$$\mathfrak{T}^+(\psi_-) = -im(1-q^{-2})\alpha\,\beta\,\psi_-. \tag{2.25}$$

Given this class of Hodge operators, the reality condition (2.16) can be extended to a more general compatibility of the action of $\mathfrak{T}^+$ with the $*$-conjugation in $\Omega_{+\mathrm{inv}}(\mathrm{SU}_q(2))$; for any left-invariant "eigenform" (2.10) of the antisymmetriser $A_+^{(k)}$ one has

$$\lambda_{\xi^*}^+\,(\mathfrak{T}^+(\xi))^* = \lambda_\xi^+\mathfrak{T}^+(\xi^*). \tag{2.26}$$

We close the analysis of the Hodge operator $\mathfrak{T}^+$ by noticing that, on the same basis, for a real and maximally hermitian contraction one has

$$(\mathfrak{T}^+)^2(\xi) = (-1)^{k(4-k)}\frac{\lambda_\xi^+}{\lambda_{\xi^*}^+}\,(\mathfrak{T}^+)^2(1)\,\xi. \tag{2.27}$$

In order to clarify this relation we can define the quantum determinant of the contraction map Γ_+ as

$$\mathrm{det}_q\,\Gamma_+ := \Gamma_+(i\,\omega_-\otimes\omega_+\otimes\omega_0\otimes\omega_z, i\,\omega_-\wedge\omega_+\wedge\omega_0\wedge\omega_z) \tag{2.28}$$

and fix (up to a sign) the scale parameter $m\in\mathbb{R}$ imposing

$$(\mathfrak{T}^+)^2(1) = \mathrm{sgn}\,\Gamma_+ \qquad\Leftrightarrow\qquad m^2(\alpha\,\beta\,\epsilon^2) = 1, \tag{2.29}$$

with $\mathrm{sgn}\,(\Gamma_+) = (\mathrm{det}_q\Gamma_+)\backslash\,|\mathrm{det}_q\Gamma_+|$. This choice allows to write the relation (2.27) as

$$(\mathfrak{T}^+)^2(\xi) = (-1)^{k(4-k)}(\mathrm{sgn}\,\Gamma_+)\frac{\lambda_\xi^+}{\lambda_{\xi^*}^+}\,\xi, \tag{2.30}$$

which generalises the classical (1.6) to the quantum setting, where the braiding σ associated to the calculus on $\mathrm{SU}_q(2)$ has the non trivial spectral decomposition given in (B.19).

2.3. **The Hodge operator $\mathfrak{T}^-$.** It is now easy to study the Hodge operator $\mathfrak{T}^-$ on $\Omega_{-\mathrm{inv}}(\mathrm{SU}_q(2))$, following the path outlined for $\mathfrak{T}^+$. Given the U(1)-coinvariant contraction map Γ in (2.13), from the second relation out of (B.25) we have

$$\mathfrak{T}^-(1) = \check\mu = \mu = \mathfrak{T}^+(1),$$
$$\mathfrak{T}^-(\check\mu) = \mathfrak{T}^+(\mu), \tag{2.31}$$

while, from the first relation out of (B.25),

$$\mathfrak{T}^-(\omega_a) = \mathfrak{T}^+(\omega_a). \tag{2.32}$$

Some algebra allows to work out the action of $\mathfrak{T}^-$ on $\Omega_{-\mathrm{inv}}^3(\mathrm{SU}_q(2))$, so to obtain the conditions of reality and hermitianity of the contraction Γ_-; for real and hermitian contraction maps one has (with $\chi_a = \check\chi_a$, recall (B.25))

$$\mathfrak{T}^-(\chi_a) = \mathfrak{T}^+(\chi_a). \tag{2.33}$$

8

Some more algebra gives then the action of $\mathfrak{T}^-$ on $\Omega^2_{-\mathrm{inv}}$ and proves indeed that the family of real and maximally hermitian contractions Γ_- for $\mathfrak{T}^-$ coincides with that obtained above by the Hodge operator $\mathfrak{T}^+$. For those contractions:

$$
\begin{aligned}
\mathfrak{T}^-(\check{\varphi}_+) &= -im\,\alpha\,\epsilon\,\check{\varphi}_+, \\
\mathfrak{T}^-(\check{\kappa}_+) &= -im\,\alpha\,\epsilon\,\check{\kappa}_+, \\
\mathfrak{T}^-(\check{\psi}_+) &= -im(q^{-2}-1)\alpha\,\beta\,\check{\psi}_+, \\
\mathfrak{T}^-(\check{\varphi}_-) &= im\,q^2\alpha\,\epsilon\,\check{\varphi}_-, \\
\mathfrak{T}^-(\check{\kappa}_-) &= im\,q^2\alpha\,\epsilon\,\check{\kappa}_-, \\
\mathfrak{T}^-(\check{\psi}_-) &= im(1-q^2)\alpha\,\beta\,\check{\psi}_-.
\end{aligned}
\tag{2.34}
$$

The Hodge operator $\mathfrak{T}^-$ corresponding to real and maximally hermitian contractions satisfies, on the left-invariant basis of $\Omega_-(\mathrm{SU}_q(2))$ given by (2.10), the same reality condition (2.26) that $\mathfrak{T}^+$ satisfies, namely

$$
\lambda^-_{\xi^*}\,(\mathfrak{T}^-(\xi))^* = \lambda^-_\xi\,\mathfrak{T}^-(\xi^*),
\tag{2.35}
$$

and its square gives, in complete analogy to (2.27),

$$
(\mathfrak{T}^-)^2(\xi) = (-1)^{k(4-k)}\frac{\lambda^-_\xi}{\lambda^-_{\xi^*}}\,(\mathfrak{T}^-)^2(1)\,\xi.
\tag{2.36}
$$

It is now immediate to check that, with

$$
\det{}_q\Gamma_- := \Gamma_-(i\,\omega_-\otimes\omega_+\otimes\omega_0\otimes\omega_z, i\,\omega_-\wedge\omega_+\wedge\omega_0\wedge\omega_z) = \det{}_q\Gamma_+
$$

(where the last equality comes from (2.31)), from the condition $(\mathfrak{T}^-)^2(1) = \mathrm{sgn}\,\Gamma_- \Leftrightarrow m^2(\alpha\,\beta\,\epsilon^2) = 1$ one can write

$$
(\mathfrak{T}^-)^2(\xi) = (-1)^{k(4-k)}(\mathrm{sgn}\,\Gamma_-)\frac{\lambda^-_\xi}{\lambda^-_{\xi^*}}\,\xi.
\tag{2.37}
$$

Although U(1)-coinvariant real and maximally hermitian contractions $\Gamma_\pm$ *coincide*, the actions (as relations (2.25) and (2.34) show) on $\Omega_{\pm\mathrm{inv}}(\mathrm{SU}_q(2))$ of the corresponding Hodge operators $\mathfrak{T}^\pm$ *do not*. The $\Omega_{-\mathrm{inv}}(\mathrm{SU}_q(2))\simeq\Omega_{+\mathrm{inv}}(\mathrm{SU}_q(2))$ isomorphism described in (B.24) and (B.25) indeed allows to recover that, with the obvious position $\det_q\Gamma_+ = \det_q\Gamma_-$ for this class of contractions,

$$
\mathfrak{T}^+\mathfrak{T}^-(\xi) = \mathfrak{T}^-\mathfrak{T}^+(\xi) = (-1)^{k(4-k)}(\mathrm{sgn}\,\Gamma)\,\xi;
\tag{2.38}
$$

this relation shows, in a quantum setting with $\sigma^2\neq 1$, the closest similarity to the classical (1.6).

The last step is to extend the Hodge operators to the whole exterior algebras $\Omega_\pm(\mathrm{SU}_q(2))$ by requiring them to be left (resp. right) $\mathcal{A}(\mathrm{SU}_q(2))$-linear. We define then *Hodge duality operators* $\star^{L,R}_\pm : \Omega^k(\mathrm{SU}_q(2)) \to \Omega^{4-k}(\mathrm{SU}_q(2))$ by

$$
\begin{aligned}
\star^L_\pm(x\,\omega) &:= x\,\mathfrak{T}^\pm(\omega), \\
\star^R_\pm(\omega\,x) &:= \mathfrak{T}^\pm(\omega)\,x,
\end{aligned}
\tag{2.39}
$$

(with $x\in\mathcal{A}(\mathrm{SU}_q(2))$ and $\omega\in\Omega_{\mathrm{inv}}(\mathrm{SU}_q(2))$) for the class of U(1)-coinvariant real and maximally hermitian contraction map Γ considered above. We summarise the results of this section as a proposition.

Proposition 2.2. *Given the Hodge operators $\mathfrak{T}^\pm$ defined in (2.11), their corresponding sets of real and maximally hermitian contractions $\Gamma_\pm$ do coincide. It is always possible to rescale (with $m\in\mathbb{R}$)*

9

the volume form $\mu = \check{\mu} = im\,\omega_- \wedge \omega_+ \wedge \omega_0 \wedge \omega_z$ such that relations (2.26), (2.30) as well as (2.35), (2.37) hold. Their extension to the Hodge duality operators moreover satisfy

$$\star_\pm^{L,R}\star_\mp^{L,R}(\phi) = (-1)^{k(4-k)}(\mathrm{sgn}\,\Gamma)\phi$$

for any $\phi \in \Omega^k(\mathrm{SU}_q(2))$.

3. Symmetric contractions and Laplacians on $\mathrm{SU}_q(2)$

For the class of real and maximally hermitian contractions one has $\det_q\Gamma = -q^2(1-q^2)^2\alpha^4$, so that $\mathrm{sgn}\,\Gamma = -1$: we recover here a phenomenon which also the formulation in [5, 6] presents. For a deeper analysis of the relations between the two formulations, following (2.3) we define a metric structure on the left-invariant part of the first order differential calculus over $\mathrm{SU}_q(2)$ as a bilinear map $\tilde{g}_q : \Omega_{\mathrm{inv}}^{\otimes 2}(\mathrm{SU}_q(2)) \to \mathbb{C}$ set by

$$\tilde{g}_q(\omega_a, \omega_b) := \Gamma(\omega_a^*, \omega_b) \tag{3.1}$$

associated to contraction maps Γ above. Relations (2.19) show that this class of metrics, that we denote by $\mathfrak{G}$, can be parametrised by $(\mathbb{R}\backslash 0) \times \mathbb{Z}_2$, that is a nonzero real parameter together with the choice of a sign. In the matrix notation (2.13),

$$\tilde{g}_{q\,ab} = \begin{pmatrix} 0 & q^2a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm(1-q^2)a \\ 0 & 0 & \pm(1-q^2)a & \pm(q^2+1)a \end{pmatrix} \tag{3.2}$$

with $a = -\alpha \neq 0$. In the formulation introduced and developed in [5, 6], Hodge duality operators acting on $\Omega(\mathrm{SU}_q(2))$ are defined from non-degenerate σ-metrics, which are maps $g_I : \Omega^{\otimes 2}(\mathrm{SU}_q(2)) \to \mathcal{A}(\mathrm{SU}_q(2))$ satisfying the conditions

- g_I is an homomorphism of the $\mathcal{A}(\mathrm{SU}_q(2))$-bimodule $\Omega^{\otimes 2}(\mathcal{A}(\mathrm{SU}_q(2)))$;
- g_I is non-degenerate;
- $g_I \circ \sigma^\pm = g_I$ (symmetry condition);
- the equality $(g_I \otimes 1) \circ \sigma_2^\pm = (1 \otimes g_I) \circ \sigma_1^\mp$ on $\Omega^{\otimes 3}(\mathrm{SU}_q(2))$ holds;

σ-metrics are called real provided $(g_I(\phi, \phi'))^* = g_I(\phi'^*, \phi^*)$, and left-covariant provided $\Delta \circ g_I = (1 \otimes g_I)\Delta_L^{(2)}$.

In order to compare these two definitions, we restrict our analysis to the set – that we denote $\mathfrak{G}_\sigma$ – of real σ-metrics given by $g_I(\omega_a, \omega_b) \in \mathbb{C}$ with ω_j left-invariant 1-forms (such are left-covariant), and which are in addition right U(1)-coinvariant, namely (using the notation of (2.12)) those satisfying $n_a + n_b \neq 0 \Rightarrow g_I(\omega_a, \omega_b) = 0$ with $\delta_R^{(1)}(\omega_a) = z^{n_a} \otimes \omega_a$.

An explicit calculation proves that $\mathfrak{G}_\sigma \subset \mathfrak{G}$: any $g_I \in \mathfrak{G}_\sigma$ coincides with a metric structure $\tilde{g}_q$ in (3.2) corresponding to the contraction with $\epsilon = (1-q^2)\alpha$, $\gamma = (1+q^2)\alpha$.

We gain a further perspective on metrics in $\mathfrak{G}_\sigma$ and $\mathfrak{G}\backslash\mathfrak{G}_\sigma$ by focussing on the spectra of the Laplacians associated to the Hodge dualities (2.39). The equalities (2.32), (2.33) allow to define operators $\Box^{L,R} : \mathcal{A}(\mathrm{SU}_q(2)) \to \mathcal{A}(\mathrm{SU}_q(2))$ whose action can be written in terms of the basic derivations (B.8), (B.10),

$$\Box^L x := -\star^L \mathrm{d} \star^L \mathrm{d}\,x = \alpha\{L_+L_- + q^2L_-L_+ \mp (1+q^2)L_zL_z \pm 2(q^2-1)L_0L_z\}\triangleright x,$$

$$\Box^R x := -\star^R \mathrm{d} \star^R \mathrm{d}\,x = \alpha\{q^2R_+R_- + R_-R_+ \mp (1+q^2)R_zR_z \pm 2(q^2-1)R_0R_z\}\triangleright x. \tag{3.3}$$

These operators are diagonal on the vector space basis $\mathcal{A}(\mathrm{SU}_q(2)) \supset \mathcal{L}_n \ni \phi_{n,J,l} = (c^{J-n/2}a^{*J+n/2})\triangleleft E^l$ with $n \in \mathbb{Z}$, $J = |n|/2 + \mathbb{N}$, $l = 0, \ldots, 2J$, and their spectrum is given in [11, §5]. We report the analysis on Laplacians $\Box^L$ in more details, being the pattern for $\Box^R$ analogue.

For a metric in $\mathfrak{G}_\sigma$ we have, from (B.8) and the (B.9) which gives the quantum Casimir of $\mathcal{U}_q(\mathfrak{su}(2))$,

$$\square^L x = \alpha\{L_+L_- + q^2 L_-L_+ + (1+q^2)L_z L_z - 2(q^2-1)L_0 L_z\} \triangleright x = 2q\,\alpha\, L_0 \triangleright x, \tag{3.4}$$

with

$$L_0 \triangleright \phi_{n,J,l} = [J][J+1]\phi_{n,J,l}.$$

The action of this Laplacian[1] reduces in the classical limit to the action of the quadratic Casimir C of the Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$. It is indeed well known that C coincides with the Laplacian corresponding to the Cartan-Killing metric on SU(2): it seems then meaningful to assume that a metric $\tilde{g}_q \in \mathfrak{G}_\sigma$ reduces in the classical limit to a Cartan-Killing metric on SU(2). Along the same path it is also immediate to recover that, although real and maximally hermitian contractions have the same signature in the quantum setting, corresponding metrics in $\mathfrak{G}_\sigma$ and $\mathfrak{G}\backslash\mathfrak{G}_\sigma$ have different signatures in the classical limit. The discrete spectrum of Laplacians corresponding to metrics in $\mathfrak{G}\backslash\mathfrak{G}_\sigma$ moreover ranges on the whole real line: although seems it meaningless to use them to model energy levels of a stable quantum system, this class of Laplacians show that the formalism developed here consistently introduce some features of a non Riemannian geometry on $\mathrm{SU}_q(2)$.

Remark 3.1. *The Laplacian* (3.4) *shows an interesting relation with the Laplacian Δ_q introduced in [1] on $\mathrm{SU}_q(2)$ in terms of the R-matrix structure of $\mathcal{A}(\mathrm{SU}_q(2))$ and without any reference to a differential calculus. If we fix $\alpha = (q^2+1)^{-1}$ in (3.4), then*

$$\Delta_q = \square^L + \left(\frac{q-q^{-1}}{q+q^{-1}}\right)^2 L_0^2 \tag{3.5}$$

as elements in $\mathcal{U}_q(\mathfrak{su}(2))$. To cast the analysis in [1] in the perspective of a general study of Dirac operators on $\mathrm{SU}_q(2)$ we refer to [10] and reference therein.

3.1. **A comparison.** We describe now how the set of metrics $\mathfrak{G}$ can be characterised by an analysis of the Hodge operators $\mathfrak{L}^\pm$ on $\Omega_{\pm inv}(\mathrm{SU}_q(2))$ defined in [11] as

$$\mathfrak{L}^\pm(\xi) := \frac{1}{\lambda_\xi^\pm} \Gamma_\pm^*(\xi, \mu). \tag{3.6}$$

This epression clearly generalises the classical (2.8). From the definition of shuffle operators one has $B_{1,N-1}^\pm = 1$, so that (with $\mu = \check\mu$)

$$\mathfrak{L}^\pm(1) = \mu,$$
$$\mathfrak{L}^\pm(\mu) = \frac{1}{\lambda_\mu} \Gamma_\pm^*(\mu, \mu),$$
$$\mathfrak{L}^\pm(\omega_a) = \mathfrak{T}^\pm(\omega_a) \tag{3.7}$$

on any left-invariant 1-form ω_a. The contraction Γ given in (2.13) is said real and hermitian provided the conditions

$$\omega_a^* \wedge L(\omega_b) = \Gamma(\omega_a, \omega_b)\mu,$$
$$L(\omega^*) = (L(\omega_a))^* \tag{3.8}$$

are satisfied. Such conditions amount to set

$$(\alpha \neq 0,\ \beta = q^2\alpha,\ \nu,\ \gamma,\ \epsilon,\ \xi = \epsilon) \in \mathbb{R}, \tag{3.9}$$

together with the implicit condition of invertibility $\Gamma_\pm(\mu, \mu) \neq 0$.

---

[1]The equality $\square^R x = \square^L x$ if $\tilde{g}_q \in \mathfrak{G}_\sigma$ is immediate to check.

The analysis of the spectra of the operators $\mathfrak{L}^{\pm}$ leads to characterise real and maximally hermitian contractions $\Gamma_{\pm}$ with respect to the Hodge operators $\mathfrak{L}^{\pm}$. Although long, straightforward calculations show that the family of real and maximally hermitian contractions for $\mathfrak{L}^{\pm}$ *does coincide* with that corresponding to $\mathfrak{T}^{\pm}$, whose parameters fullfill the conditions (2.19).

For real and maximally hermitian contractions it is further possible to prove that the action of $\mathfrak{L}^{\pm}(\xi)$ coincides with that of $\mathfrak{T}^{\pm}(\xi)$ on 2- and 3- left-invariant forms in (2.10); this is indeed sufficient to prove that operators $\mathfrak{L}^{\pm}$ satisfy the same reality conditions satisfied by $\mathfrak{T}^{\pm}$ (given in (2.26) and (2.35)),

$$\lambda_{\xi^*}^{\pm}\left(\mathfrak{L}^{\pm}(\xi)\right)^* = \lambda_{\xi}^{\pm}\mathfrak{L}^{\pm}(\xi^*). \tag{3.10}$$

For this set of contractions, one can calculate that

$$\Gamma_{\pm}(\mu,\mu) = -2(q+q^{-1})^2(q^2+1+q^{-2})\,m^2\,\alpha\,\beta\,\epsilon^2. \tag{3.11}$$

Recalling (B.23), it is then

$$\mathfrak{L}^{\pm}(\mu) \neq \mathfrak{T}^{\pm}(\mu). \tag{3.12}$$

This difference says that relations like (2.30), (2.37), (2.38) can not (for any choice of the real scale parameter m) be satisfied by the Hodge operators $\mathfrak{L}^{\pm}$. This is the reason why we do not extend them to proper Hodge dualities as in (2.39), but only use to give a different characterisation of the class of real and maximally hermitian contractions.

Remark 3.2. *A Hodge operator on $\Omega(\mathrm{S}_q^2)$, the left covariant three dimensional exterior algebra on the homogeneous quantum Podleś sphere S_q^2 induced by the $\Omega(\mathrm{SU}_q(2))$ we have considered here, has been introduced in [11] via an inner product obtained as a restriction to $\Omega(\mathrm{S}_q^2)$ of a suitable class of contractions on $\Omega(\mathrm{SU}_q(2))$.*

It is immediate to see that the condition $\Gamma(\omega_0,\omega_0) = \nu = 0$ (2.19) amounts to say that the class of real and maximally hermitian contractions on $\Omega(\mathrm{SU}_q(2))$ would induce a degenerate inner product on $\Omega(\mathrm{S}_q^2)$. The problem of defining a Hodge operator on $\Omega(\mathrm{S}_q^2)$ whose corresponding Laplacian give a spectrum equivalent to the square of the Dirac operator [2] requires further work.

Appendix A. Exterior differential calculi on Hopf algebras

This appendix recalls, from the theory of differential calculi on algebras and quantum groups, definitions and general results needed to make the description of the exterior algebras over $\mathrm{SU}_q(2)$ self consistent. A more complete analysis is in [14, 8].

Let $\mathcal{H}$ be the quantum group given by the unital $*$-Hopf algebra $\mathcal{H} = (\mathcal{H}, \Delta, \varepsilon, S)$ over $\mathbb{C}$, with $\Omega^1(\mathcal{H})$ an $\mathcal{H}$-bimodule. Given the linear map $\mathrm{d} : \mathcal{H} \to \Omega^1(\mathcal{H})$, the pair $(\Omega^1(\mathcal{H}), \mathrm{d})$ is a (first order) differential calculus over $\mathcal{H}$ if d satisfies the Leibniz rule, $\mathrm{d}(h\,h') = (\mathrm{d}h)h' + h\,\mathrm{d}h'$ for $h, h' \in \mathcal{H}$, and if $\Omega^1(\mathcal{H})$ is generated by $\mathrm{d}(\mathcal{H})$ as a $\mathcal{H}$-bimodule. It is called a $*$-calculus if there is an anti-linear involution $* : \Omega^1(\mathcal{H}) \to \Omega^1(\mathcal{H})$ such that $(h_1(\mathrm{d}h)h_2)^* = h_2^*(\mathrm{d}(h^*))h_1^*$ for any $h, h_1, h_2 \in \mathcal{H}$.

A first order differential calculus is said left covariant provided a left coaction $\Delta_L^{(1)} : \Omega^1(\mathcal{H}) \to \mathcal{H} \otimes \Omega^1(\mathcal{H})$ exists, such that $\Delta_L^{(1)}(\mathrm{d}h) = (1 \otimes \mathrm{d})\Delta(h)$ and $\Delta_L^{(1)}(h_1\,\alpha\,h_2) = \Delta(h_1)\Delta_L^{(1)}(\alpha)\Delta(h_2)$ for any $h, h_1, h_2 \in \mathcal{H}$ and $\alpha \in \Omega^1(\mathcal{H})$.

The property of right covariance is stated in complete analogy; there is in addition the notion of a bicovariant calculus, namely a calculus which is both left and right covariant and satisfying the compatibility condition:

$$(1 \otimes \Delta_R^{(1)}) \circ \Delta_L^{(1)} = (\Delta_L^{(1)} \otimes 1) \circ \Delta_R^{(1)}.$$

Left covariant calculi on $\mathcal{H}$ are in one to one correspondence with right ideals $\mathcal{Q} \subset \ker \varepsilon$. There is a left $\mathcal{H}$-modules isomorphism given by $\Omega^1(\mathcal{H}) \simeq \mathcal{H} \otimes (\ker \varepsilon / \mathcal{Q})$, which allows to recover the complex vector space $\ker \varepsilon / \mathcal{Q}$ as the set of left invariant 1-forms, namely the elements $\omega_a \in \Omega^1(\mathcal{H})$ such that

$$\Delta_L^{(1)}(\omega_a) = 1 \otimes \omega_a.$$

12

The dimension of $\ker\varepsilon/\mathcal{Q}$ is referred to as the dimension of the calculus. A left covariant first order differential calculus is a $*$-calculus if and only if $(S(Q))^* \in \mathcal{Q}$ for any $Q \in \mathcal{Q}$. If this is the case, the left coaction of $\mathcal{H}$ on $\Omega^1(\mathcal{H})$ is compatible with the $*$-structure: $\Delta_L^{(1)}(\mathrm{d}h^*) = (\Delta^{(1)}(\mathrm{d}h))^*$. Bicovariant calculi corresponds to right ideals $\mathcal{Q} \subset \ker\varepsilon$ which are in addition stable under the right adjoint coaction Ad of $\mathcal{H}$ onto itself, that is to say $\mathrm{Ad}(\mathcal{Q}) \subset \mathcal{Q}\otimes\mathcal{H}^2$. With G a compact connected and real Lie group, and $\mathcal{H} = \mathcal{A}(G)$ the standard Hopf $*$-algebra of complex valued functions defined on G, the well known natural bicovariant calculus corresponds to the choice $\mathcal{Q} = (\ker\varepsilon)^2$.

The right coaction of $\mathcal{H}$ on $\Omega^1(\mathcal{H})$ defines matrix elements

$$\Delta_R^{(1)}(\omega_a) = \sum_b \omega_b \otimes J_{ba}. \tag{A.1}$$

where $J_{ab} \in \mathcal{H}$. This matrix is invertible, since $\sum_b S(J_{ab})J_{bc} = \delta_{ac}$ and $\sum_b J_{ab}S(J_{bc}) = \delta_{ac}$. In addition one finds that $\Delta(J_{ab}) = \sum_c J_{ac} \otimes J_{cb}$ and $\varepsilon(J_{ab}) = \delta_{ab}$. It gives a basis of right invariant 1-forms, $\eta_a = \omega_b S(J_{ba})$

In order to construct an exterior algebra $\Omega(\mathcal{H})$ over the bicovariant first order differential calculus $(\Omega^1(\mathcal{H}), \mathrm{d})$ one uses a braiding map replacing the flip automorphism τ. Define the tensor product $\Omega^{\otimes k}(\mathcal{H}) = \Omega^1(\mathcal{H})\otimes_\mathcal{H}\ldots\otimes_\mathcal{H}\Omega^1(\mathcal{H})$ with k factors. There exists a unique $\mathcal{H}$-bimodule homomorphism $\sigma : \Omega^{\otimes 2}(\mathcal{H}) \to \Omega^{\otimes 2}(\mathcal{H})$ such that $\sigma(\omega \otimes \eta) = \eta \otimes \omega$ for any left invariant 1-form ω and any right invariant 1-form η. The map σ is invertible and commutes with the left coaction of $\mathcal{H}$:

$$(1 \otimes \sigma) \circ \Delta_L^{(2)} = \Delta_L^{(2)} \circ \sigma,$$

with $\Delta_L^{(2)}$ the extension of the coaction to the tensor product. There is an analogous invariance for the right coaction. Moreover, σ satisfies a braid equation. On $\Omega^{\otimes 3}(\mathcal{H})$:

$$(1 \otimes \sigma) \circ (\sigma \otimes 1) \circ (1 \otimes \sigma) = (\sigma \otimes 1) \circ (1 \otimes \sigma) \circ (\sigma \otimes 1). \tag{A.2}$$

The braiding map provides an antisymmetrizer operator $A^{(k)} : \Omega^{\otimes k}(\mathcal{H}) \to \Omega^{\otimes k}(\mathcal{H})$ for any $k \geq 2$. Consider the permutation group $P(k)$ of k elements. Any $p \in P(k)$ can be written as a product $p = \Pi_{j\in I(p)}\tau_{j_1}\cdots\tau_{j_{\natural(I(p))}}$ of $I(p)$ ($\natural(I(p))$ is the cardinality of $I(p)$) nearest neighbour transpositions $\{\tau_1,\ldots,\tau_{k-1}\}$ (i.e. elementary flips), where τ_j transposes j with $j+1$ leaving all other elements in $\{1,\ldots,k\}$ fixed. Using the braiding, set $\sigma_j : \Omega^{\otimes k}(\mathcal{H}) \to \Omega^{\otimes k}(\mathcal{H})$ as $\sigma_j := 1 \otimes \ldots \otimes \sigma \otimes \ldots \otimes 1$, where the product contains $k-1$ factors and σ occurs in the jth place. It is then immediate to associate $P(k) \ni p \mapsto \mathfrak{P} = \Pi_{j\in I(p)}\sigma_{j_1}\cdots\sigma_{j_{\natural I(p)}}$, and to define

$$A^{(k)} := \Pi_{p\in P(k)}\mathrm{sign}(p)\,\mathfrak{P}(p) \tag{A.3}$$

as the (order k) antisymmetriser operator associated to the braiding of the calculus. It is a bimodule automorphism; the Hopf ideals $\mathcal{S}_\mathcal{Q}^{(k)} = \ker A^{(k)}$ give the quotients

$$\Omega^k(\mathcal{H}) = \Omega^1(\mathcal{H})^{\otimes k}/\mathcal{S}_\mathcal{Q}^{(k)} \tag{A.4}$$

the structure of a $\mathcal{H}$-bicovariant bimodule which can be written as $\Omega^k(\mathcal{H}) = \mathrm{Range}\,A^{(k)}$. The exterior algebra is $(\Omega(\mathcal{H}) = \oplus_k\Omega^k(\mathcal{H}), \wedge)$ with the identification $\Omega^0(\mathcal{H}) = \mathcal{H}$. The exterior derivative is extended to $\Omega(\mathcal{H})$ as the only degree one derivation such that $\mathrm{d}^2 = 0$. The algebra $\Omega(\mathcal{H})$ has natural left and right $\mathcal{H}$-comodule structure, given by recursively setting

$$\Delta_L^{(k+1)}(\mathrm{d}\theta) = (1 \otimes \mathrm{d})\Delta_L^{(k)}(\theta), \qquad \Delta_R^{(k+1)}(\mathrm{d}\theta) = (\mathrm{d} \otimes 1)\Delta_R^{(k)}(\theta).$$

---

[2] Explicitly, $\mathrm{Ad} = (\mathrm{id}\otimes m)\,(\tau \otimes \mathrm{id})\,(S \otimes \Delta)\,\Delta$, with τ the flip operator, or $\mathrm{Ad}(h) = h_{(2)} \otimes \big(S(h_{(1)})h_{(3)}\big)$ in Sweedler notation.

Finally, the $*$-structure on $\Omega^1(\mathcal{H})$ is extended to an antilinear $* : \Omega(\mathcal{H}) \to \Omega(\mathcal{H})$ by $(\theta \wedge \theta')^* = (-1)^{kk'}\theta'^* \wedge \theta^*$ with $\theta \in \Omega^k(\mathcal{H})$ and $\theta' \in \Omega^{k'}(\mathcal{H})$; the exterior derivative operator satisfies the identity $(\mathrm{d}\theta)^* = \mathrm{d}(\theta^*)$.

The square of the braiding map does not in general coincide with the identity. The inverse map σ^{-1} also satisfies a braid equation as in (A.2), and provides, in analogy to (A.3), a set of antisymmetrizer operators $A_-^{(k)} : \Omega^1(\mathcal{H})^{\otimes k} \to \Omega^1(\mathcal{H})^{\otimes k}$, giving Hopf ideals $\mathcal{S}_{\mathcal{Q}_-}^{(k)} = \ker A^{(k)} = \ker A^{(k)} = \mathcal{S}_{\mathcal{Q}}$. The quotients $\Omega_-^k(\mathcal{H}) = \Omega^1(\mathcal{H})^{\otimes k}/\mathcal{S}_{\mathcal{Q}_-}^{(k)}$ describe then an exterior algebra $(\Omega_-(\mathcal{H}) = \oplus_k \Omega_-^k(\mathcal{H}), \vee)$ which is isomorphic to the $\Omega(\mathcal{H})$ considered above.

A.1. **Derivations associated to exterior differential calculi.** The tangent space of the calculus is the complex vector space of elements out of $\mathcal{H}'$ – the dual space $\mathcal{H}'$ of functionals on $\mathcal{H}$ – defined by
$$\mathcal{X}_{\mathcal{Q}} := \{L \in \mathcal{H}' \ : \ L(1) = 0, \ L(Q) = 0, \ \forall Q \in \mathcal{Q}\}.$$
There exists a unique bilinear form
$$\{\ ,\ \} : \mathcal{X}_{\mathcal{Q}} \times \Omega^1(\mathcal{H}), \qquad \{L, x\mathrm{d}y\} := \varepsilon(x)L(y), \tag{A.5}$$
giving a non-degenerate dual pairing between the vector spaces $\mathcal{X}_{\mathcal{Q}}$ and $\ker \varepsilon / \mathcal{Q}$, and then a vector space isomorphism $\mathcal{X}_{\mathcal{Q}} \simeq (\ker \varepsilon / \mathcal{Q})$. The dual space $\mathcal{H}'$ has natural left and right (mutually commuting) actions on $\mathcal{H}$:
$$L \triangleright h := h_{(1)} X(h_{(2)}), \qquad h \triangleleft L := X(h_{(1)}) h_{(2)}. \tag{A.6}$$
If the vector space $\mathcal{X}_{\mathcal{Q}}$ is finite dimensional its elements belong to the dual Hopf algebra $\mathcal{H}' \supset \mathcal{H}^o = (\mathcal{H}^o, \Delta_{\mathcal{H}^o}, \varepsilon_{\mathcal{H}^o}, S_{\mathcal{H}^o})$, defined as the largest Hopf $*$-subalgebra contained in $\mathcal{H}'$. In such a case the $*$-structures are compatible with both actions,
$$L \triangleright h^* = ((S(L))^* \triangleright h)^*, \qquad h^* \triangleleft L = (h \triangleleft (S(L))^*)^*,$$
for any $L \in \mathcal{H}^o$, $h \in \mathcal{H}$ and the exterior derivative can be written as:
$$\mathrm{d}h := \sum_a (L_a \triangleright h)\, \omega_a = \sum_a \omega_a(-S^{-1}(L_a)) \triangleright h, \tag{A.7}$$
where $\{L_a, \omega_b\} = \delta_{ab}$. The twisted Leibniz rule of derivations of the basis elements L_a is dictated by their coproduct:
$$\Delta_{\mathcal{H}^o}(L_a) = 1 \otimes L_a + \sum_b L_b \otimes f_{ba}, \tag{A.8}$$
where the $f_{ab} \in \mathcal{H}^o$ consitute an algebra representation of $\mathcal{H}$, also controlling the $\mathcal{H}$-bimodule structure of $\Omega^1(\mathcal{H})$:
$$\omega_a h = \sum_b (f_{ab} \triangleright h)\omega_b, \qquad h\omega_a = \sum_b \omega_b \left((S^{-1}(f_{ab})) \triangleright h\right), \qquad \text{for} \quad h \in \mathcal{H}. \tag{A.9}$$

APPENDIX B. THE QUANTUM GROUP $\mathrm{SU}_q(2)$

The polynomial algebra $\mathcal{A}(\mathrm{SU}_q(2))$ of the quantum group $\mathrm{SU}_q(2)$ is the unital $*$-algebra generated by elements a and c, with relations
$$ac = qca \quad ac^* = qc^*a \quad cc^* = c^*c,$$
$$a^*a + c^*c = aa^* + q^2 cc^* = 1. \tag{B.1}$$
The deformation parameter $q \in \mathbb{R}$ can be restricted without loss of generality to the interval $0 < q < 1$: in the formal limit $q \to 1$ one recovers the commutative coordinate algebra on the group manifold $\mathrm{SU}(2)$. Relations (B.1) are equivalent to the unitarity of the matrix
$$U = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix},$$

14

while the Hopf algebra structures for $\mathcal{A}(\mathrm{SU}_q(2))$ can be expressed as:

$$\Delta U = U \otimes U, \qquad S(U) = U^*, \qquad \varepsilon(U) = 1.$$

The quantum universal envelopping algebra $\mathcal{U}_q(\mathfrak{su}(2))$ is the unital Hopf $*$-algebra generated as an algebra by four elements $K^{\pm 1}, E, F$, with $KK^{-1} = 1$ and the relations:

$$K^{\pm}E = q^{\pm}EK^{\pm}, \qquad K^{\pm}F = q^{\mp}FK^{\pm}, \qquad [E, F] = \frac{K^2 - K^{-2}}{q - q^{-1}}. \tag{B.2}$$

The $*$-structure is $K^* = K$, $E^* = F$, and the Hopf algebra structure is provided by coproduct

$$\Delta(K^{\pm}) = K^{\pm} \otimes K^{\pm}, \quad \Delta(E) = E \otimes K + K^{-1} \otimes E, \quad \Delta(F) = F \otimes K + K^{-1} \otimes F,$$

while the antipode and counit are

$$S(K) = K^{-1}, \quad S(E) = -qE, \quad S(F) = -q^{-1}F$$
$$\varepsilon(K) = 1, \quad \varepsilon(E) = \varepsilon(F) = 0.$$

The quadratic element

$$C_q = \frac{qK^2 - 2 + q^{-1}K^{-2}}{(q - q^{-1})^2} + FE - \tfrac{1}{4} \tag{B.3}$$

is a quantum Casimir operator that generates the centre of $\mathcal{U}_q(\mathfrak{su}(2))$.

The Hopf $*$-algebras $\mathcal{U}_q(\mathfrak{su}(2))$ and $\mathcal{A}(\mathrm{SU}_q(2))$ are dually paired. The $*$-compatible bilinear mapping $\langle \, , \, \rangle : \mathcal{U}_q(\mathfrak{su}(2)) \times \mathcal{A}(\mathrm{SU}_q(2)) \to \mathbb{C}$ is given on the generators by

$$\langle K^{\pm}, a \rangle = q^{\mp 1/2}, \qquad \langle K^{\pm}, a^* \rangle = q^{\pm 1/2},$$
$$\langle E, c \rangle = 1, \qquad \langle F, c^* \rangle = -q^{-1}, \tag{B.4}$$

with all other couples of generators pairing to zero. This pairing is proved [8] to be non-degenerate. The algebra $\mathcal{U}_q(\mathfrak{su}(2))$ is recovered as a $*$-Hopf subalgebra in the dual algebra $\mathcal{A}(\mathrm{SU}_q(2))^o$, the largest Hopf $*$-subalgebra contained in the dual vector space $\mathcal{A}(\mathrm{SU}_q(2))'$. There are [13] $*$-compatible canonical commuting actions of $\mathcal{U}_q(\mathfrak{su}(2))$ on $\mathcal{A}(\mathrm{SU}_q(2))$:

$$h \triangleright x := x_{(1)} \left\langle h, x_{(2)} \right\rangle, \qquad x \triangleleft h := \left\langle h, x_{(1)} \right\rangle x_{(2)}.$$

Given the algebra $\mathcal{A}(\mathrm{U}(1)) := \mathbb{C}[z, z^*]/< zz^* - 1 >$, the map

$$\pi : \mathcal{A}(\mathrm{SU}_q(2)) \to \mathcal{A}(\mathrm{U}(1)), \qquad \pi \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} := \begin{pmatrix} z & 0 \\ 0 & z^* \end{pmatrix} \tag{B.5}$$

is a surjective Hopf $*$-algebra homomorphism, so that $\mathrm{U}(1)$ is a quantum subgroup of $\mathrm{SU}_q(2)$ with right coaction:

$$\delta_R := (\mathrm{id} \otimes \pi) \circ \Delta \, : \, \mathcal{A}(\mathrm{SU}_q(2)) \to \mathcal{A}(\mathrm{SU}_q(2)) \otimes \mathcal{A}(\mathrm{U}(1)). \tag{B.6}$$

This right coaction gives a decomposition

$$\mathcal{A}(\mathrm{SU}_q(2)) = \oplus_{n \in \mathbb{Z}} \mathcal{L}_n, \qquad \mathcal{L}_n := \{x \in \mathcal{A}(\mathrm{SU}_q(2)) \, : \, \delta_R(x) = x \otimes z^{-n}\}, \tag{B.7}$$

with $\mathcal{A}(\mathrm{S}_q^2) = \mathcal{L}_0$ the algebra of the standard Podleś sphere, each $\mathcal{A}(\mathrm{S}_q^2)$-bimodule $\mathcal{L}_n$ gives the set of (charge n) $\mathrm{U}(1)$-coequivariant maps for the topological quantum principal bundle $\mathcal{A}(\mathrm{S}_q^2) \hookrightarrow \mathcal{A}(\mathrm{SU}_q(2))$.

B.1. A 4D exterior algebra over the quantum group $\mathrm{SU}_q(2)$**.** This section presents, following [11], the exterior algebra over the so called $4D_+$ bicovariant calculus on $\mathrm{SU}_q(2)$, which was introduced as a first order differential calculus in [14].

The ideal $\mathcal{Q}_{\mathrm{SU}_q(2)} \subset \ker \varepsilon_{\mathrm{SU}_q(2)}$ corresponding to the $4D_+$ calculus is generated by the nine elements $\{c^2;\ c(a^*-a);\ q^2a^{*2}-(1+q^2)(aa^*-cc^*)+a^2;\ c^*(a^*-a);\ c^{*2};\ [q^2a+a^*-q^{-1}(1+q^4)]c;\ [q^2a+a^*-q^{-1}(1+q^4)](a^*-a);\ [q^2a+a^*-q^{-1}(1+q^4)]c^*;\ [q^2a+a^*-q^{-1}(1+q^4)][q^2a+a^*-(1+q^2)]\}$. One has $\mathrm{Ad}(\mathcal{Q}_{\mathrm{SU}_q(2)}) \subset \mathcal{Q}_{\mathrm{SU}_q(2)} \otimes \mathcal{A}(\mathrm{SU}_q(2))$ and $\dim(\ker \varepsilon_{\mathrm{SU}_q(2)}/\mathcal{Q}_{\mathrm{SU}_q(2)}) = 4$. The quantum tangent space is a four dimensional $\mathcal{X}_{\mathcal{Q}} \subset \mathcal{U}_q(\mathfrak{su}(2))$, with a basis given by

$$L_- = q^{\frac{1}{2}}FK^{-1}, \qquad L_z = \frac{K^{-2}-1}{q-q^{-1}}, \qquad L_+ = q^{-\frac{1}{2}}EK^{-1};$$

$$L_0 = \frac{q(K^2-1)+q^{-1}(K^{-2}-1)}{(q-q^{-1})^2} + FE = \frac{q(K^{-2}-1)+q^{-1}(K^2-1)}{(q-q^{-1})^2} + EF. \tag{B.8}$$

The vector L_0 belongs to the centre of $\mathcal{U}_q(\mathfrak{su}(2))$, and differs from the quantum Casimir (B.3) by a constant term,

$$C_q = L_0 + [\tfrac{1}{2}]^2 - \tfrac{1}{4}. \tag{B.9}$$

The differential d $: \mathcal{A}(\mathrm{SU}_q(2)) \mapsto \Omega^1(\mathrm{SU}_q(2))$ is written for any $x \in \mathcal{A}(\mathrm{SU}_q(2))$ as (with $R_a = -S^{-1}(L_a)$)

$$\mathrm{d}x = \sum_a (L_a \triangleright x)\omega_a = \sum_a \omega_a(R_a \triangleright x), \qquad \text{with } R_a = -S^{-1}(L_a) \tag{B.10}$$

on the dual basis of left invariant forms $\omega_a \in \Omega^1(\mathrm{SU}_q(2))$, which are

$$\begin{aligned}
\omega_- &= c^*\mathrm{d}a^* - qa^*\mathrm{d}c^*, \\
\omega_+ &= a\mathrm{d}c - qc\mathrm{d}a, \\
\omega_z &= a^*\mathrm{d}a + c^*\mathrm{d}c - (a\mathrm{d}a^* + q^2c\mathrm{d}c^*), \\
\omega_0 &= (1+q)^{-1}\rho^{-1}[a^*\mathrm{d}a + c^*\mathrm{d}c + q(a\mathrm{d}a^* + q^2c\mathrm{d}c^*)],
\end{aligned} \tag{B.11}$$

where $\rho = [\tfrac{1}{2}][\tfrac{3}{2}]$. This differential calculus reduces in the classical limit to the standard three-dimensional bicovariant calculus on $\mathrm{SU}(2)$, since one has $\omega_0 \to 0$ for $q \to 1$. Since $(S(\mathcal{Q}_{\mathrm{SU}_q(2)}))^* \subset \mathcal{Q}_{\mathrm{SU}_q(2)}$, the $*$-structure on $\mathcal{A}(\mathrm{SU}_q(2))$ is consistently extended to an antilinear $*$-structure on $\Omega^1(\mathrm{SU}_q(2))$, such that $(\mathrm{d}x)^* = \mathrm{d}(x^*)$ for any $x \in \mathcal{A}(\mathrm{SU}_q(2))$. For the basis of left invariant 1-forms it is

$$\omega_-^* = -\omega_+, \qquad \omega_z^* = -\omega_z, \qquad \omega_0^* = -\omega_0. \tag{B.12}$$

The construction of the exterior algebras $\Omega(\mathrm{SU}_q(2)) \simeq \Omega_-(\mathrm{SU}_q(2))$ is given via the braiding map $\sigma : \Omega^{\otimes 2}(\mathrm{SU}_q(2)) \to \Omega^{\otimes 2}(\mathrm{SU}_q(2))$ and its inverse.

$$\sigma^{\pm}(\omega_- \otimes \omega_-) = \omega_- \otimes \omega_-, \tag{B.13}$$

$$\sigma \begin{pmatrix} \omega_- \otimes \omega_0 \\ \omega_0 \otimes \omega_- \\ \omega_- \otimes \omega_z \\ \omega_z \otimes \omega_- \end{pmatrix} = \begin{pmatrix} 1-q^2 & 1 & 0 & 0 \\ q^2 & 0 & 0 & 0 \\ 1+q^2 & 0 & 0 & 1 \\ -1-q^{-2} & 0 & q^{-2} & 1-q^{-2} \end{pmatrix} \begin{pmatrix} \omega_- \otimes \omega_0 \\ \omega_0 \otimes \omega_- \\ \omega_- \otimes \omega_z \\ \omega_z \otimes \omega_- \end{pmatrix},$$

$$\sigma^{-1} \begin{pmatrix} \omega_- \otimes \omega_0 \\ \omega_0 \otimes \omega_- \\ \omega_- \otimes \omega_z \\ \omega_z \otimes \omega_- \end{pmatrix} = \begin{pmatrix} 0 & q^{-2} & 0 & 0 \\ 1 & 1-q^{-2} & 0 & 0 \\ 0 & 1+q^2 & 1-q^2 & q^2 \\ 0 & -1-q^{-2} & 1 & 0 \end{pmatrix} \begin{pmatrix} \omega_- \otimes \omega_0 \\ \omega_0 \otimes \omega_- \\ \omega_- \otimes \omega_z \\ \omega_z \otimes \omega_- \end{pmatrix}, \tag{B.14}$$

$$
\sigma \begin{pmatrix} \omega_z \otimes \omega_0 \\ \omega_0 \otimes \omega_z \\ \omega_z \otimes \omega_z \\ \omega_- \otimes \omega_+ \\ \omega_+ \otimes \omega_- \end{pmatrix} = \begin{pmatrix} -(q - q^{-1})^2 & 1 & 0 & -(q - q^{-1})^2 & (q - q^{-1})^2 \\ 1 & 0 & 0 & 0 & 0 \\ q^2 - q^{-2} & 0 & 1 & q^2 - q^{-2} & -(q^2 - q^{-2}) \\ -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \omega_z \otimes \omega_0 \\ \omega_0 \otimes \omega_z \\ \omega_z \otimes \omega_z \\ \omega_- \otimes \omega_+ \\ \omega_+ \otimes \omega_- \end{pmatrix},
$$

$$
\sigma^{-1} \begin{pmatrix} \omega_z \otimes \omega_0 \\ \omega_0 \otimes \omega_z \\ \omega_z \otimes \omega_z \\ \omega_- \otimes \omega_+ \\ \omega_+ \otimes \omega_- \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -(q - q^{-1})^2 & 0 & -(q - q^{-1})^2 & (q - q^{-1})^2 \\ 0 & q^2 - q^{-2} & 1 & q^2 - q^{-2} & -(q^2 - q^{-2}) \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \omega_z \otimes \omega_0 \\ \omega_0 \otimes \omega_z \\ \omega_z \otimes \omega_z \\ \omega_- \otimes \omega_+ \\ \omega_+ \otimes \omega_- \end{pmatrix},
$$
$$(\text{B.15})$$

$$
\sigma^{\pm}(\omega_0 \otimes \omega_0) = \omega_0 \otimes \omega_0 \tag{B.16}
$$

$$
\sigma \begin{pmatrix} \omega_+ \otimes \omega_0 \\ \omega_0 \otimes \omega_+ \\ \omega_+ \otimes \omega_z \\ \omega_z \otimes \omega_+ \end{pmatrix} = \begin{pmatrix} 1 - q^{-2} & 1 & 0 & 0 \\ q^{-2} & 0 & 0 & 0 \\ -1 - q^{-2} & 0 & 0 & 1 \\ 1 + q^2 & 0 & q^2 & 1 - q^2 \end{pmatrix} \begin{pmatrix} \omega_+ \otimes \omega_0 \\ \omega_0 \otimes \omega_+ \\ \omega_+ \otimes \omega_z \\ \omega_z \otimes \omega_+ \end{pmatrix},
$$

$$
\sigma^{-1} \begin{pmatrix} \omega_+ \otimes \omega_0 \\ \omega_0 \otimes \omega_+ \\ \omega_+ \otimes \omega_z \\ \omega_z \otimes \omega_+ \end{pmatrix} = \begin{pmatrix} 0 & q^2 & 0 & 0 \\ 1 & 1 - q^2 & 0 & 0 \\ 0 & -1 - q^{-2} & 1 - q^{-2} & q^{-2} \\ 0 & 1 + q^2 & 1 & 0 \end{pmatrix} \begin{pmatrix} \omega_+ \otimes \omega_0 \\ \omega_0 \otimes \omega_+ \\ \omega_+ \otimes \omega_z \\ \omega_z \otimes \omega_+ \end{pmatrix}, \tag{B.17}
$$

$$
\sigma(\omega_+ \otimes \omega_+) = \omega_+ \otimes \omega_+. \tag{B.18}
$$

The spectral resolution of the braiding gives
$$
(\sigma^{\pm} + 1)(\sigma^{\pm} + q^2)(\sigma^{\pm} + q^{-2}) = 0; \tag{B.19}
$$
it allows to give the spectral resolution of the antisymmetriser operators [12]. With $\Omega^k(\mathrm{SU}_q(2)) \ni \omega_{a_1} \wedge \ldots \wedge \omega_{a_k} = A^{(k)}(\omega_{a_1} \otimes \ldots \otimes \omega_{a_k})$, one has from [11] that:

$$
\begin{aligned}
\varphi_{\pm} &:= \omega_{\mp} \wedge \omega_0, & A^{(2)}(\varphi_{\pm}) &= (1 + q^{\pm 2})\varphi_{\pm} \\
\kappa_{\pm} &:= \omega_{\pm} \wedge \omega_0 \mp (1 - q^{\pm 2})\omega_{\pm} \wedge \omega_z, & A^{(2)}(\kappa_{\pm}) &= (1 + q^{\pm 2})\kappa_{\pm} \\
\psi_{\pm} &:= \omega_0 \wedge \omega_z + (1 - q^{\pm 2})\omega_- \wedge \omega_+, & A^{(2)}(\psi_{\pm}) &= (1 + q^{\pm 2})\psi_{\pm}
\end{aligned} \tag{B.20}
$$

on a basis of the 6-dimensional vector space of left-invariant 2-forms, with $\varphi_{\pm}^* = \varphi_{\mp}$, $\kappa_{\pm}^* = \kappa_{\mp}$, $\psi_{\pm}^* = \psi_{\mp}$. On 3-forms it is

$$
\begin{aligned}
\chi_- &:= \omega_+ \wedge \omega_0 \wedge \omega_z, & \chi_+ &:= \omega_- \wedge \omega_0 \wedge \omega_z \\
\chi_0 &:= \omega_- \wedge \omega_+ \wedge \omega_z, & \chi_z &:= \omega_- \wedge \omega_+ \wedge \omega_0,
\end{aligned} \tag{B.21}
$$

$$
A^{(3)}(\chi_a) = 2(1 + q^2 + q^{-2})\chi_a \tag{B.22}
$$

for $a = -, +, z, 0$, with $\chi_-^* = -q^{-2}\chi_+$, $\chi_0^* = \chi_0$ and $\chi_z^* = \chi_z$. The exterior algebra corresponding to this calculus is 4-dimensional; the $\mathcal{A}(\mathrm{SU}_q(2))$-bimodule $\Omega^4(\mathrm{SU}_q(2))$ is 1 dimensional. The action of the antisymmetriser on the left invariant basis element $\vartheta = \omega_- \wedge \omega_+ \wedge \omega_z \wedge \omega_0$ gives

$$
A^{(4)}(\vartheta) = 2(q^4 + 2q^2 + 6 + 2q^{-2} + q^{-4})\vartheta. \tag{B.23}
$$

The analysis of the exterior algebra $\Omega_-(\mathrm{SU}_q(2))$, introduced in appendix §A and constructed via the "inverse braiding" σ^{-1} map, shows a similar pattern. With $\Omega_-^k(\mathrm{SU}_q(2)) \ni \omega_{a_1} \vee \ldots \vee \omega_{a_k} =$

$A_-^{(k)}(\omega_{a_1} \otimes \ldots \otimes \omega_{a_k})$, one has

$$\check{\varphi}_\pm := \omega_\mp \vee \omega_0 = q^{\mp 2}\varphi_\pm, \qquad A_-^{(2)}(\check{\varphi}_\pm) = (1+q^{\mp 2})\check{\varphi}_\pm$$

$$\check{\kappa}_\pm := \omega_\pm \vee \omega_0 \mp (1-q^{\pm 2})\omega_\pm \vee \omega_z = q^{\mp 2}\kappa_\pm, \qquad A_-^{(2)}(\check{\kappa}_\pm) = (1+q^{\mp 2})\check{\kappa}_\pm$$

$$\check{\psi}_\pm := \omega_0 \vee \omega_z + (1-q^{\pm 2})\omega_- \vee \omega_+ = q^{\mp 2}\psi_\pm, \qquad A_-^{(2)}(\check{\psi}_\pm) = (1+q^{\mp 2})\check{\psi}_\pm \qquad \text{(B.24)}$$

on 2-forms, with $\check{\varphi}_\pm^* = \check{\varphi}_\mp$, $\check{\kappa}_\pm^* = \check{\kappa}_\mp$, $\check{\psi}_\pm^* = \check{\psi}_\mp$; while on 3- and 4-forms, with the same notations used in (B.21),

$$\check{\chi}_a = \chi_a, \qquad A_-^{(3)}(\check{\chi}_a) = 2(1+q^2+q^{-2})\check{\chi}_a$$

$$\check{\vartheta} = \omega_- \vee \omega_+ \vee \omega_0 \vee \omega_z = \vartheta, \qquad A_-^{(4)}(\check{\vartheta}) = 2(q^4 + 2q^2 + 6 + 2q^{-2} + q^{-4})\check{\vartheta}. \qquad \text{(B.25)}$$

These relations show that the anti-linear $*$-structure does not commute with the action of the antisymmetrisers $A_\pm^{(2)}$.

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